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T. R. Seifullin

## Homology of the Koszul complex of a system of polynomial equations

(Presented by Corresponding Member of the NAS of Ukraine A. A. Letichevsky)

*Explicit complex morphism of a dual complex to the Koszul complex into the Koszul complex is constructed. If the number of common roots of polynomials is finite in an algebraically closed field, then this mapping is a homotopic equivalence, thus explicit duality of the Koszul complex is obtained.*

In the present paper it has generalized author's results of [1,2] on the whole Koszul complex of polynomials of a system of polynomial equations, for the arbitrary number of polynomials, greater equal to the number of variables. In result it has constructed explicit complex morphism of a complex dual to the Koszul complex into the Koszul complex. In the case of the finite number of common roots of polynomials in algebraically closed field, this mapping is a homotopic equivalence, thus it is obtained explicit duality for the Koszul complex.

**Koszul complexes.** Let  $\mathbf{R}$  be a commutative ring,  $f(x) = (f_1(x), \dots, f_s(x)) \in \mathbf{R}[x]^s$  be polynomials in  $x = (x_1, \dots, x_n)$  with coefficients in a commutative ring  $\mathbf{R}$ ,  $\forall i : f_i(x) \equiv 0$  be a system of polynomial equations. Represent an element  $a = (a_1, \dots, a_s) \in \mathbf{R}^s$  in the form  $a\hat{f}_* = \sum_i a_i \hat{f}_*^i$ , and  $\forall i$  denote by  $\hat{f}_i$  a linear functional defined on  $\mathbf{R}^s$ , assigning  $(a_1, \dots, a_s) \mapsto a_i$ , where  $\hat{f} = (\hat{f}_1, \dots, \hat{f}_s)$  and  $\hat{f}_* = (\hat{f}_*^1, \dots, \hat{f}_*^s)$  be systems of variables. Any functional is represented in the form  $\hat{f}b = \sum_j \hat{f}_j b^j$  with the action  $a\hat{f}_* \cdot \hat{f}b = \sum_i a_i \hat{f}_*^i \cdot \sum_j \hat{f}_j b^j = \sum_i a_i b^i = ab$ . Then the element  $f(x) = (f_1(x), \dots, f_s(x))$  is represented in the form  $f(x)\hat{f}_* = \sum_i f_i(x)\hat{f}_*^i$ . Denote by  $(f(x))_x$  the ideal of the ring  $\mathbf{R}[x]$  generated by polynomials  $f(x)$ .

For any graded module  $\mathbf{P} = \bigoplus_r \mathbf{P}_r$ , denote  $\mathbf{P}_\times = \bigcup_r \mathbf{P}_r$ . If  $p \in \mathbf{P}_r$ , then we will write  $|p| = r$ . Put  $\forall i : |\hat{f}_*^i| = -1$ ,  $|\hat{f}_i| = 1$ . Let  $\Lambda(\hat{f}_*) = \bigoplus_r \Lambda_{-r}(\hat{f}_*)$  be a Grassmann algebra, i.e. an associative algebra with 1, free generated by elements

$(\hat{f}_*^1, \dots, \hat{f}_*^s)$  with relations  $\left\{ \left( \sum_i f_i \hat{f}_*^i \right) \left( \sum_i f_i \hat{f}_*^i \right) = 0 \right\}$ , where  $f = (f_1, \dots, f_s)$  are mutually commuting variables,  $\Lambda_{-r}(\hat{f}_*)$  is the set of sums of products of  $r$  elements in  $(\hat{f}_*)$ .

Denote by  $\Lambda_r(\hat{f}_*)^*$  the module dual to  $\Lambda_{-r}(\hat{f}_*)$ , i. e. the set of  $\mathbf{R}$ -linear maps  $\Lambda_{-r}(\hat{f}_*) \rightarrow \mathbf{R}$ . Define a product in  $\Lambda(\hat{f}_*)^*$  by

$$\forall c_1(\hat{f}), c_2(\hat{f}) \in \Lambda_{\times}(\hat{f}_*)^* : \forall a(\hat{f}_*) \in \Lambda_{\times}(\hat{f}_*) : a(\hat{f}_*) \cdot c_1(\hat{f}) \cdot c_2(\hat{f}) = a(\hat{f}'_* + \hat{f}''_*) \cdot c_1(\hat{f}') \cdot c_2(\hat{f}''),$$

where  $\forall a_1(\hat{f}_*), a_2(\hat{f}_*) \in \Lambda_{\times}(\hat{f}_*) : a_2(\hat{f}''_*) \cdot a_1(\hat{f}'_*) \cdot c_1(\hat{f}') \cdot c_2(\hat{f}'') = \left( a_1(\hat{f}_*) \cdot c_1(\hat{f}) \right) \left( a_2(\hat{f}_*) \cdot c_2(\hat{f}) \right)$ ,  $\hat{f}' \simeq \hat{f}'' \simeq \hat{f}$  are equivalent systems of variables. Then  $\Lambda_r(\hat{f}_*)^* \simeq \Lambda_r(\hat{f})$ .

Define  $\mathbf{C}_{-r}(x, \hat{f}_*) = \mathbf{R}[x] \otimes \Lambda_{-r}(\hat{f}_*)$ ,  $\mathbf{C}(x, \hat{f}_*) = \bigoplus_r \mathbf{C}_{-r}(x, \hat{f}_*)$ , then  $f(x) \hat{f}_* = \sum_i f_i(x) \hat{f}_*^i \in \mathbf{C}_{-1}(x, \hat{f}_*)$ . Define  $\mathbf{C}(x, \hat{f}) = \bigoplus_r \mathbf{C}_r(x, \hat{f})$ , where  $\mathbf{C}_r(x, \hat{f})$  is the set of  $\mathbf{R}$ -linear maps of  $\Lambda_{-r}(\hat{f}_*)$  into  $\mathbf{R}[x]$ , which are equivalent to skew symmetric  $\mathbf{R}$ -polylinear forms on  $\mathbf{R}^s \simeq \mathbf{R} \otimes (\hat{f}_*)$  into  $\mathbf{R}[x]$ . It holds  $\mathbf{C}_r(x, \hat{f}) \simeq \mathbf{R}[x] \otimes \Lambda_r(\hat{f})$ .

For  $c(x, \hat{f}) \in \mathbf{C}_r(x, \hat{f})$ , if  $r \geq 1$  denote by  $\partial [c(x, \hat{f})]$  such element  $\in \mathbf{C}_{r-1}(x, \hat{f})$ , that  $\forall a(x, \hat{f}_*) \in \mathbf{C}_{-r+1}(x, \hat{f}_*) : a(x, \hat{f}_*) \cdot \partial [c(x, \hat{f})] = a(x, \hat{f}_*) \cdot f(x) \hat{f}_* \cdot c(x, \hat{f})$ , if  $r = 0$  denote  $\partial [c(x, \hat{f})] = 0$ . Then  $\forall c(x, \hat{f}) \in \mathbf{C}_r(x, \hat{f}) : \partial [\partial [c(x, \hat{f})]] = 0$ . Denote by  $(\mathbf{C}(x, \hat{f}); \partial) = (\mathbf{C}(x, \hat{f}); \hat{f} \mapsto f(x))$  a complex, which is called the Koszul complex,  $\mathbf{Z}(x, \hat{f}) = \{c \in \mathbf{C}(x, \hat{f}) | \partial [c] = 0\}$ ,  $\mathbf{B}(x, \hat{f}) = \{\partial [c] | c \in \mathbf{C}(x, \hat{f})\}$ . From  $\partial^2 = 0$  it follows that  $\mathbf{B}_r(x, \hat{f}) \subseteq \mathbf{Z}_r(x, \hat{f})$ . Denote by  $\mathbf{H}_r(x, \hat{f}) = \mathbf{Z}_r(x, \hat{f}) / \mathbf{B}_r(x, \hat{f})$ .

It hold

$$\begin{aligned} \forall c_1, c_2 \in \mathbf{C}_{\times}(x, \hat{f}) : \partial [c_1 \cdot c_2] &= \partial [c_1] \cdot c_2 + (-1)^{|c_1|} c_1 \cdot \partial [c_2], \\ \mathbf{Z}(x, \hat{f}) \cdot \mathbf{Z}(x, \hat{f}) &\subseteq \mathbf{Z}(x, \hat{f}), \mathbf{B}(x, \hat{f}) \cdot \mathbf{Z}(x, \hat{f}) \subseteq \mathbf{B}(x, \hat{f}), \mathbf{Z}(x, \hat{f}) \cdot \mathbf{B}(x, \hat{f}) \subseteq \mathbf{B}(x, \hat{f}). \end{aligned}$$

Let  $v = (v_1, \dots, v_m)$  and  $\hat{h} = (\hat{h}_1, \dots, \hat{h}_t)$  be systems of variables,  $h(v) = (h_1(v), \dots, h_t(v)) \in \mathbf{R}[v]^t$ , and  $(\mathbf{C}(v, \hat{h}); \partial) = (\mathbf{C}(v, \hat{h}); \hat{h} \mapsto h(v))$  be the Koszul complex of a system of polynomials  $h(v)$ . Denote by  $(\mathbf{C}(v, \hat{h}, x_*, \hat{f}_*); \partial) = (\mathbf{C}(v, \hat{h}, x_*, \hat{f}_*); \hat{f} \mapsto f(x), \hat{h} \mapsto h(v))$  the complex of  $\mathbf{R}$ -linear maps  $\mathbf{C}(x, \hat{f}) \rightarrow \mathbf{C}(v, \hat{h})$ , in which the boundary operator defined as following:  $\forall c \in \mathbf{C}_{\times}(x, \hat{f}) : \forall a \in \mathbf{C}_{\times}(v, \hat{h}, x_*, \hat{f}_*) : \partial [a] \cdot c = \partial [a \cdot c] - (-1)^{|a|} a \cdot \partial [c]$ , where  $\mathbf{C}_r(v, \hat{h}, x_*, \hat{f}_*) = \{a \in \mathbf{C}(v, \hat{h}, x_*, \hat{f}_*) | \forall r' : a \cdot \mathbf{C}_{r'+r}(x, \hat{f}) \subseteq \mathbf{C}_{r'+r}(v, \hat{h})\}$ . Denote by  $\mathbf{Z}(v, \hat{h}, x_*, \hat{f}_*) = \{a \in \mathbf{C}(v, \hat{h}, x_*, \hat{f}_*) | \partial [a] = 0\}$ ,  $\mathbf{B}(v, \hat{h}, x_*, \hat{f}_*) = \{\partial [a] | a \in \mathbf{C}(v, \hat{h}, x_*, \hat{f}_*)\}$ ,  $\mathbf{H}(v, \hat{h}, x_*, \hat{f}_*) = \mathbf{Z}(v, \hat{h}, x_*, \hat{f}_*) / \mathbf{B}(v, \hat{h}, x_*, \hat{f}_*)$ .

Elements of  $\mathbf{Z}(v, \hat{h}, x_*, \hat{f}_*)$  are called complex morphisms, elements of  $\mathbf{B}(v, \hat{h}, x_*, \hat{f}_*)$  are maps homotopic to zero.

If  $v = ()$ ,  $h(v) = ()$  and  $\hat{h} = ()$ , then  $(\mathbf{C}(x_*, \hat{f}_*); \partial) = (\mathbf{C}(v, \hat{h}, x_*, \hat{f}_*); \partial)$  is the complex dual to the Koszul complex  $(\mathbf{C}(x, \hat{f}); \partial)$ . If  $x = ()$ ,  $f(x) = ()$  and  $\hat{f} = ()$ , then  $(\mathbf{C}(v, \hat{h}, x_*, \hat{f}_*); \partial) = (\mathbf{C}(v, \hat{h}); \partial)$ .

Let  $(\mathbf{C}(w, \hat{g}, v_*, \hat{h}_*); \partial) = (\mathbf{C}(w, \hat{g}, v_*, \hat{h}_*); \hat{g} \mapsto g(w), \hat{h} \mapsto h(v))$  be a complex of  $\mathbf{R}$ -linear maps  $\mathbf{C}(v, \hat{h}) \rightarrow \mathbf{C}(w, \hat{g})$ , then

$$\begin{aligned} \forall a \in \mathbf{C}_{\times}(w, \hat{g}, v_*, \hat{h}_*) : \forall b \in \mathbf{C}_{\times}(v, \hat{h}, x_*, \hat{f}_*) : \partial [a \cdot b] &= \partial [a] \cdot b + (-1)^{|a|} a \cdot \partial [b], \\ \mathbf{Z}(w, \hat{g}, v_*, \hat{h}_*) \cdot \mathbf{Z}(v, \hat{h}, x_*, \hat{f}_*) &\subseteq \mathbf{Z}(w, \hat{g}, v_*, \hat{h}_*), \end{aligned}$$

$$\begin{aligned}\mathbf{B}(w, \widehat{g}, v_*, \widehat{h}_*) \cdot \mathbf{Z}(v, \widehat{h}, x_*, \widehat{f}_*) &\subseteq \mathbf{B}(w, \widehat{g}, x_*, \widehat{f}_*), \\ \mathbf{Z}(w, \widehat{g}, v_*, \widehat{h}_*) \cdot \mathbf{B}(v, \widehat{h}, x_*, \widehat{f}_*) &\subseteq \mathbf{B}(w, \widehat{g}, x_*, \widehat{f}_*).\end{aligned}$$

Let  $a(v, \widehat{h}), b(v, \widehat{h}) \in \mathbf{C}_\times(v, \widehat{h})$  and  $(-1)^{|a|} = 1, (-1)^{|b|} = -1$ , denote by  $\mathbf{1}_{(x, \widehat{f})}(a(v, \widehat{h}), b(v, \widehat{h}))$  the map of the form

$$\mathbf{C}(x, \widehat{f}) \ni c(x, \widehat{f}) \mapsto \mathbf{1}_{(x, \widehat{f})}(a(v, \widehat{h}), b(v, \widehat{h})) \cdot c(x, \widehat{f}) = c(a(v, \widehat{h}), b(v, \widehat{h})) \in \mathbf{C}(v, \widehat{h}).$$

Define the product

$$\begin{aligned}\mathbf{C}_\times(x, \widehat{f}, v_*, \widehat{h}_*) \times \mathbf{C}_\times(x, \widehat{f}, w_*, \widehat{g}_*) \ni (c, c') &\mapsto c \cdot c' \in \mathbf{C}_\times(x, \widehat{f}, v_*, \widehat{h}_*, w_*, \widehat{g}_*) : \\ \forall a \in \mathbf{C}_\times(v, \widehat{h}) : \forall a' \in \mathbf{C}_\times(w, \widehat{g}) : c \cdot c' \cdot a \cdot a' &= (-1)^{|a||c'|} (c \cdot a) \cdot (c' \cdot a').\end{aligned}$$

Then

$$\partial[c \cdot c'] = \partial[c] \cdot c' + (-1)^{|c|} c \cdot \partial[c'].$$

Let  $\mathbf{C}_\times(x, \widehat{f}, x_*, \widehat{f}_*, w, \widehat{g}, v_*, \widehat{h}_*) \ni c(x, \widehat{f}, x_*, \widehat{f}_*, w, \widehat{g}, v_*, \widehat{h}_*) =$   
 $p(w, \widehat{g}, v_*, \widehat{h}_*) \cdot a(x_*, \widehat{f}_*) \cdot a'(x, \widehat{f}) \in \mathbf{C}_\times(w, \widehat{g}, v_*, \widehat{h}_*) \cdot \mathbf{C}_\times(x_*, \widehat{f}_*) \cdot \mathbf{C}_\times(x, \widehat{f}),$   
denote by  $\bigcap_{(x, \widehat{f})} c(x, \widehat{f}, x_*, \widehat{f}_*, w, \widehat{g}, v_*, \widehat{h}_*) = p(w, \widehat{g}, v_*, \widehat{h}_*) \cdot \left( a(x_*, \widehat{f}_*) \cdot a'(x, \widehat{f}) \right),$   
denote by  $\bigcap_{(x, \widehat{f})} c(x, \widehat{f}, x_*, \widehat{f}_*, w, \widehat{g}, v_*, \widehat{h}_*)$  element of  $\mathbf{C}_\times(x_*, \widehat{f}_*, w, \widehat{g}, v_*, \widehat{h}_*)$  such that  $\forall b(x, \widehat{f}) \in$   
 $\mathbf{C}_\times(x, \widehat{f}) : \bigcap_{(x, \widehat{f})} c(x, \widehat{f}, x_*, \widehat{f}_*, w, \widehat{g}, v_*, \widehat{h}_*) \cdot b(x, \widehat{f}) = p(w, \widehat{g}, v_*, \widehat{h}_*) \cdot \left( a(x_*, \widehat{f}_*) \cdot a'(x, \widehat{f}) b(x, \widehat{f}) \right)$

An exponential determinant we call

$$\det \begin{vmatrix} bc & B\widehat{f}_* \\ \widehat{h}C & 0 \end{vmatrix} = \bigcap_{\widehat{p}} (B^l \widehat{f}_* + b^l \widehat{p}_*) \cdot \dots \cdot (B^1 \widehat{f}_* + b^1 \widehat{p}_*) \cdot (\widehat{h}C_1 + \widehat{p}c_1) \cdot \dots \cdot (\widehat{h}C_m + \widehat{p}c_m),$$

where  $\forall k : B^k \widehat{f}_* + b^k \widehat{p}_* \in \Lambda_{-1}(\widehat{f}_*, \widehat{p}_*), \forall k : \widehat{h}C_k + \widehat{p}c_k \in \Lambda_1(\widehat{h}, \widehat{p}), \widehat{f}, \widehat{h}, \widehat{p}$  are collections of anticommuting variables.

**Difference Jacobian.** Let  $f(x) = (f_1(x), \dots, f_s(x)) \in \mathbf{R}[x]^s$  be polynomials in  $x = (x_1, \dots, x_n)$  with coefficients in a commutative ring  $\mathbf{R}$ . Let  $y \simeq x, x - y = (x_1 - y_1, \dots, x_n - y_n) \in \mathbf{R}[x, y]^n$ . Consider the complex  $(\mathbf{C}(x, y, \widehat{f}_x, \widehat{f}_y, \widehat{u}); \partial) = (\mathbf{C}(x, y, \widehat{f}_x, \widehat{f}_y, \widehat{u}); \widehat{f}_x \mapsto f(x), \widehat{f}_y \mapsto f(y), \widehat{u} \mapsto (x - y))$ . To shorten notations we will write  $p_x = (x, \widehat{f}_x)$  and  $p_*^x = (x_*, \widehat{f}_*^x)$ .

Define  $\Delta_{(x, \widehat{u})}(x, y, \widehat{u}) \cdot c(x, \widehat{u}) = c(x, \widehat{u}) - c(y, \widehat{u})$ .

**Lemma.** *There exists  $\nabla_{(x', \widehat{u}')} (x, y, \widehat{u}) \in \mathbf{C}(x'_*, \widehat{u}'_*, x, y, \widehat{u})$  such that*

$$\Delta_{(x', \widehat{u}')} (x, y, \widehat{u}) = \partial [\nabla_{(x', \widehat{u}')} (x, y, \widehat{u})] \in \mathbf{B}(x'_*, \widehat{u}'_*, x, y, \widehat{u}),$$

*the operator  $\nabla_{(x', \widehat{u}')} (x, y, \widehat{u})$  is called a difference homotopy operator.*

**Proof.** Set

$$\begin{aligned}\Delta_{(x, \widehat{u})}^k (x, y, \widehat{u}) \cdot c(x, \widehat{u}) &= c(y_1, \widehat{0}, \dots, y_{k-1}, \widehat{0}, x_k, \widehat{u}_k, x_{k+1}, \widehat{u}_{k+1}, \dots, x_n, \widehat{u}_n) - \\ &- c(y_1, \widehat{0}, \dots, y_{k-1}, \widehat{0}, y_k, \widehat{0}, x_{k+1}, \widehat{u}_{k+1}, \dots, x_n, \widehat{u}_n).\end{aligned}$$

Then  $\Delta = \sum_k \Delta^k = \sum_k \partial \left[ \frac{\widehat{u}_k}{x_k - y_k} \right] \cdot \Delta^k = \sum_k \partial \left[ \frac{\widehat{u}_k}{x_k - y_k} \cdot \Delta^k \right] = \partial \left[ \sum_k \frac{\widehat{u}_k}{x_k - y_k} \cdot \Delta^k \right]$ , since

$$\partial [\Delta^k] = 0.$$

It hold  $\partial[\Delta] = 0$ , and  $\Delta.\Delta = \Delta$ . Then  $\partial[\Delta.\nabla] = \Delta.\partial[\nabla] = \Delta.\Delta = \Delta$ , hence, if  $\nabla$  is a difference homotopy, then  $\Delta.\nabla$  is also a difference homotopy, moreover,  $\Delta.(\Delta.\nabla) = \Delta.\nabla$ . A difference homotopy  $\nabla$ , for which  $\Delta.\nabla = \nabla$ , we call reduced. Let  $\nabla'$  and  $\nabla''$  be two reduced difference homotopy, then  $\partial[\nabla' - \nabla''] = \Delta - \Delta = 0$ , hence,  $\nabla' - \nabla'' \in \mathbf{Z}$ . Further  $\nabla' - \nabla'' = \Delta.(\nabla' - \nabla'') = \partial[\nabla].(\nabla' - \nabla'') = \partial[\nabla.(\nabla' - \nabla'')] \in \mathbf{B}$ , hence, a reduced difference homotopy is uniquely determined, up to homotopy. If  $c(x, \hat{u}) \in \mathbf{Z}(x, \hat{u})$ , and  $\nabla', \nabla''$  are two reduced difference homotopy, then  $\nabla'.c - \nabla''.c = \partial[\nabla.(\nabla' - \nabla'')].c = \partial[\nabla.(\nabla' - \nabla'').c] \in \mathbf{B}$ , i. e. a reduced difference homotopy of an element  $\in \mathbf{Z}(x, \hat{u})$  is uniquely determined, up to addend in  $\mathbf{B}(x, y, \hat{u})$ .

For a polynomial  $F(x) \in \mathbf{R}[x] = \mathbf{C}_0(x, \hat{u}) = \mathbf{Z}_0(x, \hat{u})$  denote by  $\hat{u}\nabla F(x, y) = \sum_k \hat{u}_k \nabla^k F(x, y) = \nabla_{(x, \hat{u})}(x, y, \hat{u}).F(x)$ , then the element  $\hat{u}\nabla F(x, y)$  is a reduced difference homotopy of  $F(x)$ , since  $\Delta_{(x, \hat{u})}(x, y, \hat{u}).\hat{u}\nabla F(x, y) = \hat{u}\nabla F(x, y) - \hat{0}\nabla F(y, y) = \hat{u}\nabla F(x, y)$ , hence,  $\hat{u}\nabla F(x, y)$  is uniquely determined in  $\mathbf{H}(x, y, \hat{u})$ .

A difference Jacobian is called

$$J(p_x, p_y) = \det \left\| \begin{array}{c} \nabla f(x, y) \\ \hat{f}_x - \hat{f}_y \end{array} \right\| = \top_{\hat{u}} \det \| -\hat{u}_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u}\nabla f(x, y) \|.$$

**Lemma.**

$$1. \partial \left[ \det \left\| \begin{array}{c} \nabla f(x, y) \\ \hat{f}_x - \hat{f}_y \end{array} \right\| \right] = 0, \text{ i. e. } \det \left\| \begin{array}{c} \nabla f(x, y) \\ \hat{f}_x - \hat{f}_y \end{array} \right\| \in \mathbf{Z}(x, y, \hat{f}_x, \hat{f}_y).$$

2. A difference Jacobian is uniquely determined in  $\mathbf{H}(x, y, \hat{f}_x, \hat{f}_y)$ , independently of the choice of  $\nabla f(x, y)$ .

**Proof 1.**

$$\partial [\det \| -\hat{u}_* \|] = -\sum_k (x_k - y_k) \hat{u}_*^k \cdot \det \| -\hat{u}_* \| = 0,$$

$$\partial [\det \| \hat{f}_x - \hat{f}_y - \hat{u}\nabla f(x, y) \|] = \partial [\prod_i (\hat{f}_{i,x} - \hat{f}_{i,y} - \hat{u}\nabla f_i(x, y))] = 0, \text{ since}$$

$$\forall i : \partial [\hat{f}_{i,x} - \hat{f}_{i,y} - \hat{u}\nabla f_i(x, y)] = f_i(x) - f_i(y) - (x - y)\nabla f_i(x, y) = 0.$$

$$\text{Hence, } \partial \left[ \det \left\| \begin{array}{c} \nabla f(x, y) \\ \hat{f}_x - \hat{f}_y \end{array} \right\| \right] = \partial \left[ \top_{\hat{u}} \det \| -\hat{u}_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u}\nabla f(x, y) \| \right] = 0.$$

**Proof 2.** Since  $\forall i : \hat{u}\nabla f_i(x, y)$  is uniquely determined in  $\mathbf{H}$ , then  $\hat{f}_{i,x} - \hat{f}_{i,y} - \hat{u}\nabla f_i(x, y)$  is uniquely determined in  $\mathbf{H}$ . Hence,

$$\det \left\| \begin{array}{c} \nabla f(x, y) \\ \hat{f}_x - \hat{f}_y \end{array} \right\| = \top_{\hat{u}} \det \| -\hat{u}_* \| \prod_i (\hat{f}_{i,x} - \hat{f}_{i,y} - \hat{u}\nabla f_i(x, y)) \text{ is uniquely determined in } \mathbf{H}, \text{ since}$$

$$\forall i : \partial [\hat{f}_{i,x} - \hat{f}_{i,y} - \hat{u}\nabla f_i(x, y)] = 0 \text{ and } \partial [\det \| -\hat{u}_* \|] = 0.$$

$$\text{Define } \det \left\| \begin{array}{c} \nabla f(x, y) \quad -\hat{u}_* \\ \hat{f}_x - \hat{f}_y \quad 0 \end{array} \right\| = \top_{\hat{u}'} \det \| -\hat{u}_* - \hat{u}'_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u}'\nabla f(x, y) \|. \text{ Then}$$

$$\partial \left[ \det \left\| \begin{array}{c} \nabla f(x, y) \quad -\hat{u}_* \\ \hat{f}_x - \hat{f}_y \quad 0 \end{array} \right\| \right] = \partial \left[ \top_{\hat{u}'} \det \| -\hat{u}_* - \hat{u}'_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u}'\nabla f(x, y) \| \right] = 0.$$

**Theorem.**

$$\begin{aligned} & \det \left\| \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} \right\| \left( \mathbf{1}_{(z, \hat{f}_z)}(x, \hat{f}_x) - \mathbf{1}_{(z, \hat{f}_z)}(y, \hat{f}_y) \right) = \\ & = \partial \left[ \begin{array}{c} \top \\ \hat{x}' \end{array} \begin{array}{c} \top \\ \hat{u}' \end{array} \begin{array}{c} \top \\ \hat{u} \end{array} \det \left\| \begin{array}{cc} \frac{\nabla f(x, y)}{-\hat{f}_x + \hat{f}_y} & \hat{u}_* \\ & 0 \end{array} \right\| \nabla_{(x', \hat{u}')} (x, y, \hat{u}) \cdot \mathbf{1}_{(z, \hat{f}_z)}(x', \hat{f}_y + \hat{u}' \nabla f(x', y)) \right]. \end{aligned}$$

**Proof.**

$$\begin{aligned} & \det \left\| \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} \right\| \left( \mathbf{1}_{(z, \hat{f}_z)}(x, \hat{f}_x) - \mathbf{1}_{(z, \hat{f}_z)}(y, \hat{f}_y) \right) = \\ & = \begin{array}{c} \top \\ \hat{u} \end{array} \det \| -\hat{u}_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u} \nabla f(x, y) \| \left( \mathbf{1}_{(z, \hat{f}_z)}(x, \hat{f}_x) - \mathbf{1}_{(z, \hat{f}_z)}(y, \hat{f}_y) \right) = \\ & = \begin{array}{c} \top \\ \hat{u} \end{array} \det \| -\hat{u}_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u} \nabla f(x, y) \| \left( \mathbf{1}_{(z, \hat{f}_z)}(x, \hat{f}_y + \hat{u} \nabla f(x, y)) - \mathbf{1}_{(z, \hat{f}_z)}(y, \hat{f}_y) \right) = \\ & = \begin{array}{c} \top \\ \hat{u}' \end{array} \begin{array}{c} \top \\ \hat{u} \end{array} \det \| -\hat{u}_* - \hat{u}'_* \| \det \| \hat{f}_x - \hat{f}_y - \hat{u}' \nabla f(x, y) \| \cdot \\ & \quad \cdot \left( \mathbf{1}_{(z, \hat{f}_z)}(x, \hat{f}_y + \hat{u}' \nabla f(x, y)) - \mathbf{1}_{(z, \hat{f}_z)}(y, \hat{f}_y) \right) = \\ & = \begin{array}{c} \top \\ \hat{u} \end{array} \det \left\| \begin{array}{cc} \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} & -\hat{u}_* \\ & 0 \end{array} \right\| \left( \mathbf{1}_{(z, \hat{f}_z)}(x, \hat{f}_y + \hat{u} \nabla f(x, y)) - \mathbf{1}_{(z, \hat{f}_z)}(y, \hat{f}_y) \right) = \\ & = \begin{array}{c} \top \\ \hat{x}' \end{array} \begin{array}{c} \top \\ \hat{u}' \end{array} \begin{array}{c} \top \\ \hat{u} \end{array} \det \left\| \begin{array}{cc} \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} & -\hat{u}_* \\ & 0 \end{array} \right\| \partial \left[ \nabla_{(x', \hat{u}')} (x, y, \hat{u}) \right] \cdot \mathbf{1}_{(z, \hat{f}_z)}(x', \hat{f}_y + \hat{u}' \nabla f(x', y)) = \\ & = \partial \left[ \begin{array}{c} \top \\ \hat{x}' \end{array} \begin{array}{c} \top \\ \hat{u}' \end{array} \begin{array}{c} \top \\ \hat{u} \end{array} \det \left\| \begin{array}{cc} \frac{\nabla f(x, y)}{-\hat{f}_x + \hat{f}_y} & \hat{u}_* \\ & 0 \end{array} \right\| \nabla_{(x', \hat{u}')} (x, y, \hat{u}) \cdot \mathbf{1}_{(z, \hat{f}_z)}(x', \hat{f}_y + \hat{u}' \nabla f(x', y)) \right]. \end{aligned}$$

Denote

$$T(p_*^z, p_x, p_y) = \begin{array}{c} \top \\ \hat{x}' \end{array} \begin{array}{c} \top \\ \hat{u}' \end{array} \begin{array}{c} \top \\ \hat{u} \end{array} \det \left\| \begin{array}{cc} \frac{\nabla f(x, y)}{-\hat{f}_x + \hat{f}_y} & \hat{u}_* \\ & 0 \end{array} \right\| \nabla_{(x', \hat{u}')} (x, y, \hat{u}) \cdot \mathbf{1}_{(z, \hat{f}_z)}(x', \hat{f}_y + \hat{u}' \nabla f(x', y)).$$

**Duality.**

**Theorem-definition 1.** *J-map we call a map*

$$\mathbf{C}(x_*, \hat{f}_*^x) \ni c(x_*, \hat{f}_*^x) \mapsto \begin{array}{c} \top \\ (y, \hat{f}_y) \end{array} \det \left\| \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} \right\| c(y_*, \hat{f}_*^y) \in \mathbf{C}(x, \hat{f}_x)$$

1. *J-map is a complex morphism:*

$$\begin{array}{c} \top \\ p_y \end{array} \partial [J(p_x, p_y)] c(p_*^y) = 0.$$

2. *J-map is uniquely determined, up to homotopy, independently of the choice of  $\nabla f(x, y)$ :*

$$\begin{array}{c} \top \\ p_y \end{array} J(p_x, p_y) c(p_*^y) - \begin{array}{c} \top \\ p_y \end{array} J'(p_x, p_y) c(p_*^y) = \begin{array}{c} \top \\ p_y \end{array} \partial [S(p_x, p_y)] c(p_*^y).$$

3. *J-map is a homotopy  $\mathbf{C}(x, \hat{f}_x)$ -skewlinear:*

$$\left( \begin{array}{c} \top \\ p_y \end{array} J(p_x, p_y) c(p_*^y) \right) a(p_x) - \begin{array}{c} \top \\ p_y \end{array} J(p_x, p_y) \left( \begin{array}{c} \perp \\ p_y \end{array} c(p_*^y) a(p_y) \right) = \begin{array}{c} \top \\ p_y \end{array} \begin{array}{c} \top \\ p_z \end{array} \partial [T(p_*^z, p_x, p_y)] c(p_*^y) a(p_z).$$

**Theorem-definition 2.** *J-product is called a map*

$$\mathbf{C}(x_*, \hat{f}_*^x) \times \mathbf{C}(y_*, \hat{f}_*^y) \ni \left( c_1(x_*, \hat{f}_*^x), c_2(y_*, \hat{f}_*^y) \right) \mapsto$$

$$\perp_{(x, \hat{f}_x)} c_1(x_*, \hat{f}_*^x) \top_{(y, \hat{f}_y)} \det \left\| \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} \right\| c_2(y_*, \hat{f}_*^y) \in \mathbf{C}(x_*, \hat{f}_*^x).$$

1. *J-product is a complex bimorphism:*

$$\perp_{p_x} c_1(p_*^x) \top_{p_y} \partial [J(p_x, p_y)] c_2(p_*^y) = 0.$$

2. *J-product is uniquely determined up to homotopy, independently of the choice of  $\nabla f(x, y)$ :*

$$\perp_{p_x} c_1(p_*^x) \top_{p_y} J(p_x, p_y) c_2(p_*^y) - \perp_{p_x} c_1(p_*^x) \top_{p_y} J'(p_x, p_y) c_2(p_*^y) = \perp_{p_x} c_1(p_*^x) \top_{p_y} \partial [S(p_x, p_y)] c_2(p_*^y).$$

3. *J-product is homotopy skewcommutative:*

$$\perp_{p_x} c_1(p_*^x) \top_{p_y} J(p_x, p_y) c_2(p_*^y) - (-1)^{|c_1|' |c_2|'} \perp_{p_x} c_2(p_*^x) \top_{p_y} J(p_x, p_y) c_1(p_*^y) =$$

$$= \top_{p_y p_z} c_1(p_*^y) \partial [C(p_*^x, p_y, p_z)] c_2(p_*^z),$$

where  $|c_1|' = |c_1| + |J|$  and  $|c_2|' = |c_2| + |J|$ .

**Theorem 3.** *If the image of  $\mathbf{H}(x_*, \hat{f}_*^x)$  under the map  $J$  include  $1 \in \mathbf{H}(x, \hat{f}_x)$ , i. e. if*

$$\exists e(x_*, \hat{f}_*^x) \in \mathbf{Z}(x_*, \hat{f}_*^x) : \top_{(y, \hat{f}_y)} \det \left\| \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} \right\| e(y_*, \hat{f}_*^y) = 1 + \partial [t(x, \hat{f}_x)],$$

then the map

$$\mathbf{C}(x, \hat{f}_x) \ni c(x, \hat{f}_x) \mapsto \perp_{(x, \hat{f}_x)} e(x_*, \hat{f}_*^x) c(x, \hat{f}_x) \in \mathbf{C}(x_*, \hat{f}_*^x)$$

is homotopy inverse to the map  $J$ , i. e.

$$1) \forall c(p_x) \in \mathbf{C}(p_x) : c(p_x) - \top_{p_y} J(p_x, p_y) \left( \perp_{p_y} e(p_*^y) c(p_y) \right) = \top_{p_y} \partial [R(p_*^y, p_x)] c(p_y),$$

$$2) \forall c(p_*^x) \in \mathbf{C}(p_*^x) : c(p_*^x) - \perp_{p_x} e(p_*^x) \left( \top_{p_y} J(p_x, p_y) c(p_*^y) \right) = \top_{p_y} \partial [L(p_*^x, p_y)] c(p_*^y),$$

and inverse to the map  $J$  in  $\mathbf{H}(x, \hat{f}_x) \rightarrow \mathbf{H}(x_*, \hat{f}_*^x)$ , i. e.

$$3) \forall c(p_x) \in \mathbf{Z}(p_x) : c(p_x) - \top_{p_y} J(p_x, p_y) \left( \perp_{p_y} e(p_*^y) c(p_y) \right) = \partial \left[ \top_{p_y} R(p_*^y, p_x) c(p_y) \right],$$

$$4) \forall c(p_*^x) \in \mathbf{Z}(p_*^x) : c(p_*^x) - \perp_{p_x} e(p_*^x) \left( \top_{p_y} J(p_x, p_y) c(p_*^y) \right) = \partial \left[ \top_{p_y} L(p_*^x, p_y) c(p_*^y) \right];$$

5)  $\mathbf{H}_0(x, \hat{f}_x)$  and  $\mathbf{H}_{-s+n}(x_*, \hat{f}_*^x)$  are finitely generated as modules over  $\mathbf{R}$ .

**Theorem 4.** *If  $\mathbf{H}_0(x, \hat{f}_x) = \mathbf{R}[x]/(f(x))_x$  is a finitely generated module over  $\mathbf{R}$ , then*

$$\exists e(x_*, \hat{f}_*^x) \in \mathbf{Z}(x_*, \hat{f}_*^x) : \top_{(y, \hat{f}_y)} \det \left\| \frac{\nabla f(x, y)}{\hat{f}_x - \hat{f}_y} \right\| e(y_*, \hat{f}_*^y) = 1 + \partial [t(x, \hat{f}_x)]$$

and it hold the statements of theorem 3.

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*V. M. Glushkov Institute of Cybernetics of the NAS of Ukraine, Kiev*

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*E-mail: timur\_sf@mail.ru*